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## Noncommutative Two Dimensional BF Model

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**Abstract :** We consider the noncommutative extension of the BF theory in two spacetime dimensions. We show that the introduction of the noncommutative parameter  $\theta_{\mu\nu}$ , already at first order in the analytical sector, induces infinitely many terms in the quantum extension of the model. This clashes with the commonly accepted rules of QFT, and we believe that this problem is not peculiar to this particular model, but it might concern the noncommutative extension of any ordinary quantum field theory obtained via the Moyal prescription. A detailed study of noncommutative anomalies is also presented.

**Keywords:** Noncommutative field theory, Algebraic renormalization, BRS quantization

**PACS Nos:** 11.10.Gh Renormalization, 11.10.Nx Noncommutative field theory, 11.15.-q Gauge field theories.

# 1 Introduction

The literature on noncommutative field theory models has rapidly grown to a considerable size, and yet there is no clear cut recipe of how one should proceed to analyze such models in particular for what concerns their renormalizability. One of the main issues which arise, is whether the noncommutative model, with  $\theta_{\mu\nu}$  as the noncommutative parameter, has a smooth (analytical)  $\theta_{\mu\nu} \rightarrow 0$  limit or not. In the first case we have a corresponding standard, commutative model, which can be treated with the well established machinery of local quantum field theory, and it is tempting to apply it also to the noncommutative model [1, 2, 3, 4]. In the second case, the model we are looking at is intrinsically noncommutative, and we loose both analyticity in  $\theta_{\mu\nu}$  and locality [5, 6].

However, we would like to emphasize that in any approach the theory should be defined by the functional equations which encode its symmetry content and all the additional information needed to analyze both the stability and the anomaly issues. Of course those tools are normally applied in an environment where locality, power counting and the Quantum Action Principle [7, 8, 9, 10, 11] also hold true; we might call it a traditional reference frame. In the study of the noncommutative extensions of a standard quantum field theory model, is a matter of personal choice how much we should adhere to this traditional reference frame.

In this paper we would like to probe where we are led to by analyzing the analytical  $\theta_{\mu\nu}$  sector of the two dimensional BF model [12, 13, 14], using the entire machinery of the traditional reference frame [15]. Our idea is that the study of noncommutative models beginning from a commutative one, looking only at the sector analytical in the  $\theta^{\mu\nu}$  tensor, is of interest since it can provide hints to new and unexpected features of the noncommutative theory. Here we provide an example of problems which arise in carrying out the process of an analytical  $\theta^{\mu\nu}$  expansion. In other words, we limit ourselves to the sector of the theory in which the UV properties are controlled by the planar diagrams [1, 16]. Our aim is to show that this sector, in which the Quantum Action Principle should reign, does not admit a consistent quantum extension.

When attempting to define a noncommutative quantum field theory [17] and wishing also to arrive at a formulation which allows explicit amplitude computation, one is faced with the problem of choosing a precise form for the noncommutative product. One of the most popular choices is the Groenewold-Moyal product [18, 19] which is implemented with a simple exponential formula and needs the introduction of an antisymmetric constant tensor  $\theta^{\mu\nu}$  having the dimensions of an inverse mass squared. It is commonly accepted that this procedure leads to a well defined noncommutative theory if the commutative model we begin with is sound. We shall show that this is not always the case by providing a counterexample.

A reliable noncommutative extension of an ordinary model should be based on the functional identities encoding the symmetries, on locality and power counting, just as it happens in the commutative case. This procedure, in the standard case, leads to the stability and anomaly analysis i.e. the model is perturbatively renormalizable if the classical action is the most general local functional compatible with the above constraints (stability) and the symmetries are not broken by the radiative corrections (anomaly) [15].

The paper is organized as follows. To fix the notations, in Section 2 we briefly recall the functional equations (BRS identity, Landau gauge, ghost equation and vector supersymmetry) which form a closed algebraic structure and completely define, together with locality and power counting, the commutative model [14]. In Section 3 we deform the classical theory by means of the anticommuting constant parameter  $\theta_{\mu\nu}$ , introduced by means of the substitution of the ordinary product with the Groenewold-Moyal star product. In [20] we have showed that the defining equations remain exactly the same we have in the commutative case. Of course this is not so for the classical action which acquires, at the first order in  $\theta^{\mu\nu}$ , a local contribution with canonical dimension equal to four and coupled to  $\theta^{\mu\nu}$  itself. In Section 4 we are led to consider the stability problem of the classical action to first order in  $\theta^{\mu\nu}$ , *i.e.* we completely characterize the counterterm, and conclude that it contains a countable infinity of terms linked to two classes of coupling constants. Finally, in Section 5, we also analyzed the anomaly problem and found that, at least to first order in  $\theta^{\mu\nu}$ , the model is anomaly free. We draw our conclusions in the final Section 6.

## 2 The commutative model

The action of ordinary, commutative,  $BF$  model in two spacetime dimensions is [12, 13, 14]

$$S_{inv} = -\frac{1}{2} \text{Tr} \int d^2x \, \epsilon^{\mu\nu} F_{\mu\nu} \phi = -\frac{1}{2} \int d^2x \, \epsilon^{\mu\nu} F_{\mu\nu}^a \phi^a \quad (2.1)$$

where  $\epsilon^{\mu\nu}$  is the two-dimensional completely antisymmetric Levi-Civita tensor,  $F_{\mu\nu} = F_{\mu\nu}^a T^a = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$  is the field strength, and  $A_\mu(x) = A_\mu^a(x) T^a$  and  $\phi(x) = \phi^a(x) T^a$  are the gauge field and a scalar field respectively, belonging to the adjoint representation of the gauge group, assumed to be  $U(n)$ , whose generators  $T^a$  ( $a = 1, \dots, n^2$ ) obey

$$\text{Tr} (T^a T^b) = \delta^{ab} \quad (2.2)$$

$$T^a T^b = \frac{i}{2} f^{abc} T^c + \frac{1}{2} d^{abc} T^c \quad (2.3)$$

$f^{abc}$  and  $d^{abc}$  being the antisymmetric and symmetric Gell-Mann tensors, respectively. We will comment on this choice of the gauge group later.

The action (2.1) is invariant under the following infinitesimal gauge transformations, with  $\Lambda(x) = \Lambda^a(x)T^a$  as infinitesimal gauge parameter

$$\delta_g A_\mu = D_\mu \Lambda = \partial_\mu \Lambda - i[A_\mu, \Lambda] \quad (2.4)$$

$$\delta_g \phi = -i[\phi, \Lambda] \quad (2.5)$$

As usual, the quantization of the model proceeds with the introduction of a set of quantum fields  $(c^a(x), \bar{c}^a(x), b^a(x))$ , playing the role of ghost, antighost and Lagrange multiplier fields, respectively, by means of which the gauge fixing term reads, in the Landau gauge,

$$S_{gf} = \int d^2x \left( b^a \partial^\mu A_\mu^a - \bar{c}^a \partial^\mu (D_\mu c)^a \right) \quad (2.6)$$

The transformations (2.4)-(2.5) are no more a symmetry of the gauge fixed action  $S = S_{inv}[A] + S_{gf}[A, c, \bar{c}, b]$ , which is indeed invariant under the BRS transformations

$$\begin{aligned} sA_\mu &= D_\mu c \\ s\phi &= -i[\phi, c] \\ sc &= ic^2 \\ s\bar{c} &= b \\ sb &= 0 \end{aligned} \quad (2.7)$$

The BRS operator  $s$  is characterized by the two fundamental properties of being nilpotent

$$s^2 = 0 \quad (2.8)$$

and of being a symmetry of the theory

$$sS = 0 \quad (2.9)$$

The latter point is most easily verified once it is noticed that the gauge fixing term  $S_{gf}$  can be written as a BRS cocycle

$$S_{gf} = s \int d^2x \bar{c}^a \partial^\mu A_\mu^a \quad (2.10)$$

Since the BRS transformations (2.7) are nonlinear, external fields  $A^{*a\mu}(x)$ ,  $\phi^{*a}(x)$  and  $c^{*a}(x)$  must be coupled to the nonlinear BRS variations through

$$S_{ext} = \int d^2x \left( A^{*a\mu}(sA_\mu^a) + \phi^{*a}(s\phi^a) + c^{*a}(sc^a) \right) \quad (2.11)$$

in order to be able to write for the total classical action

$$\Sigma = S_{inv} + S_{gf} + S_{ext} \quad (2.12)$$

a Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = \text{Tr} \int d^2x \left( \frac{\delta \Sigma}{\delta A^{*\mu}} \frac{\delta \Sigma}{\delta A_\mu} + \frac{\delta \Sigma}{\delta \phi^*} \frac{\delta \Sigma}{\delta \phi} + \frac{\delta \Sigma}{\delta c^*} \frac{\delta \Sigma}{\delta c} + b \frac{\delta \Sigma}{\delta \bar{c}} \right) = 0 . \quad (2.13)$$

The action  $S$  is topological, since it does not depend on the spacetime metric  $g_{\mu\nu}$ . In other words, only the gauge fixing term of the action contributes to the energy-momentum tensor, which therefore is an exact BRS cocycle

$$T_{\mu\nu} \equiv \frac{\delta S}{\delta g^{\mu\nu}} = s \Lambda_{\mu\nu} \quad (2.14)$$

for some integrated local functional  $\Lambda_{\mu\nu}$ , which can be easily calculated from the expression of the classical action  $S$ . Now, as it has been remarked for the first time in [21], the highly non trivial observation that both  $T_{\mu\nu}$  and  $\Lambda_{\mu\nu}$  are conserved, underlies the existence of an additional symmetry of the action. Indeed, the conservation relation

$$\partial^\nu \Lambda_{\mu\nu} = \text{contact terms} \quad (2.15)$$

once integrated, directly represents the Ward identity for the vector symmetry

$$\begin{aligned} \delta_\mu A_\nu &= 0 \\ \delta_\mu \phi &= \epsilon_{\mu\nu} \partial^\nu \bar{c} \\ \delta_\mu c &= A_\mu \\ \delta_\mu \bar{c} &= 0 \\ \delta_\mu b &= \partial_\mu \bar{c} \end{aligned} \quad (2.16)$$

The existence of such a linear, vector symmetry is peculiar to topological field theories, and it is called vector *supersymmetry* due to the following algebra, formed by the operators  $s$  and  $\delta_\mu$

$$\begin{aligned} s^2 &= 0 \\ \{\delta_\mu, \delta_\nu\} &= 0 \\ \{\delta_\mu, s\} &= \partial_\mu + \text{eqs of motion} \end{aligned} \quad (2.17)$$

which, closing on shell on the spacetime translations, describe a superalgebra of the Wess-Zumino type. Once the source term (2.11) is introduced, the algebra (2.17) closes off shell, and the Ward identity for the vector supersymmetry becomes

$$W_\mu \Sigma = \Delta_\mu \quad (2.18)$$

where

$$W_\mu = \text{Tr} \int d^2x \left( \epsilon_{\mu\nu} (\partial^\nu \bar{c} + A^{*\nu}) \frac{\delta}{\delta \phi} + \epsilon_{\mu\nu} \phi^* \frac{\delta}{\delta A_\nu} + A_\mu \frac{\delta}{\delta c} + \partial_\mu \bar{c} \frac{\delta}{\delta b} - c^* \frac{\delta}{\delta A^{*\mu}} \right) \quad (2.19)$$

and  $\Delta_\mu$  is a breaking linear in the quantum fields, and therefore purely classic

$$\Delta_\mu = \text{Tr} \int d^2x \ (-A^{*\nu}\partial_\mu A_\nu - \phi^*\partial_\mu\phi + c^*\partial_\mu c + \epsilon_{\mu\nu}\phi^*\partial^\nu\bar{c}) \quad (2.20)$$

In addition, as for any other - commutative - gauge field, the choice of the Landau gauge guarantees the existence of an additional constraint on the classical action  $\Sigma$ : the “ghost” equation [22]

$$\mathcal{G}^a\Sigma = \int d^2x \left( \frac{\delta}{\delta c^a} + f^{abc}\bar{c}^b\frac{\delta}{\delta b^c} \right) \Sigma = \Delta^a, \quad (2.21)$$

where  $\Delta^a$  is again a classical breaking

$$\Delta^a = \int d^2x \ f^{abc} \left( A^{*b\mu} A_\mu^c + \phi^{*b} \phi^c - c^{*b} c^c \right). \quad (2.22)$$

The following constraints

1. the Slavnov-Taylor identity (2.13);
2. the supersymmetry (2.18);
3. the ghost equation (2.21)
4. the Landau gauge condition

$$\frac{\delta\Sigma}{\delta b^a} = \partial^\mu A_\mu^a \quad (2.23)$$

fully characterize the theory, both at the classical and at the quantum level. As showed in [14], the classical action  $\Sigma$  (2.12) is the most general one compatible with the whole set of constraints, which in turn are free of anomalies.

In other words, the ordinary, commutative BF theory in two spacetime dimensions is renormalizable to all orders of perturbation theory. More than this, we know that this theory, like any other topological quantum field theory, is finite, namely no radiative corrections are allowed by the symmetries characterizing the classical theory.

### 3 The noncommutative model

The usual and generally accepted behavior to proceed towards the noncommutative extension of any commutative field theory, is simply to substitute the ordinary product between fields with the Moyal “star” product [18, 19]

$$\phi(x)\psi(x) \longrightarrow \phi(x) * \psi(x) \equiv \lim_{y \rightarrow x} \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x\partial_\nu^y\right) \phi(x)\psi(y), \quad (3.1)$$

where  $\theta^{\mu\nu}$  is a rank-two antisymmetric matrix which controls the noncommutative nature of spacetime coordinates

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \quad (3.2)$$

The choice of  $U(n)$  as gauge group is motivated by the request of having gauge-valued noncommutative fields, which would not be the case for a generic else non abelian gauge group, for (2.3) acquires an additional term

$$T^a T^b = \frac{1}{2n} \delta^{ab} + \frac{i}{2} f^{abc} T^c + \frac{1}{2} d^{abc} T^c \quad (3.3)$$

As a consequence of the absence of the central term  $\frac{1}{2n} \delta^{ab}$ , the gauge group  $U(n)$  is closed under the star product, while, for instance,  $SU(n)$  does not give rise to any gauge group on the noncommutative plane.

In two spacetime dimensions, the commutator (3.2) is Lorentz invariant. This implies that  $\theta^{\mu\nu}$  is proportional to the Levi-Civita tensor  $\epsilon^{\mu\nu}$

$$\theta^{\mu\nu} = \frac{1}{\theta^2} \epsilon^{\mu\nu} \quad (3.4)$$

where  $\theta^2$  is the dimensional parameter induced by the commutation relation (3.2). The property (3.4) is tightly related to the two dimensional spacetime.

We will not bother the reader with the motivations for such a generalization of ordinary spacetime, and of its consequences on field theory, for which an excellent literature exists [17, 16].

In this paper, we simply investigate the consequences of the introduction in the theory of  $\theta^{\mu\nu}$  as a new ingredient in an otherwise ordinary gauge field theory. Under this respect,  $\theta^{\mu\nu}$  is no more than an antisymmetric constant parameter with negative (minus two) mass dimensions, coupling to fields to form monomials. In particular, in the example of two dimensional BF theory we ask if the deformation induced by the presence of the noncommutative parameter  $\theta^{\mu\nu}$  is compatible with the basic features of quantum field theory, namely

- stability of the classical action
- absence of anomalies

The field strength becomes

$$\begin{aligned} F_{\mu\nu}(\theta) &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig(A_\mu * A_\nu - A_\nu * A_\mu) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]_* \end{aligned} \quad (3.5)$$

and, at the first order in  $\theta$

$$F_{\mu\nu}^a(\theta) = F_{\mu\nu}^a(0) + \frac{1}{2} d^{abc} \theta^{\alpha\beta} (\partial_\alpha A_\mu^b)(\partial_\beta A_\nu^c) \quad (3.6)$$

Analogously, the covariant derivative, at the first order in  $\theta$ , reads

$$(D_\mu \Lambda)^a(\theta) = (D_\mu \Lambda)^a(0) + \frac{1}{2} d^{abc} \theta^{\alpha\beta} (\partial_\alpha A_\mu^b)(\partial_\beta \Lambda^c) \quad (3.7)$$

The invariant, gauge-fixed,  $\theta$ -dependent action is

$$\begin{aligned} S^{(\theta)} &= \text{Tr} \int d^2x \left( -\frac{1}{2} \epsilon^{\mu\nu} F_{\mu\nu}(\theta) * \phi + s^{(\theta)}(\bar{c} \partial A) \right) \\ &= S + \frac{1}{2} d^{abc} \theta^{\alpha\beta} \int d^2x \left( -\frac{1}{2} \epsilon^{\mu\nu} \phi^a (\partial_\alpha A_\mu^b)(\partial_\beta A_\nu^c) + (\partial_\mu \bar{c}^a)(\partial_\alpha A_\mu^b)(\partial_\beta c^c) \right) \\ &\quad + O(\theta^2) \end{aligned} \quad (3.8)$$

where  $S$  is the gauge-fixed commutative invariant action, and  $s^{(\theta)}$  is the noncommutative BRS operator, which, at first order in  $\theta$ , is

$$\begin{aligned} s^{(\theta)} \phi^a &= s\phi^a + \frac{1}{2} d^{abc} \theta^{\alpha\beta} (\partial_\alpha \phi^b)(\partial_\beta c^c) \\ s^{(\theta)} A_\mu^a &= sA_\mu^a + \frac{1}{2} d^{abc} \theta^{\alpha\beta} (\partial_\alpha A_\mu^b)(\partial_\beta c^c) \\ s^{(\theta)} c^a &= sc^a - \frac{1}{4} d^{abc} \theta^{\alpha\beta} (\partial_\alpha c^b)(\partial_\beta c^c) \\ s^{(\theta)} \bar{c}^a &= s\bar{c}^a \\ s^{(\theta)} b^a &= sb^a \end{aligned} \quad (3.9)$$

It is highly nontrivial that the two basic features of the BRS operator (2.8) and (2.9) are conserved when the ordinary product is deformed, via the introduction of the  $\theta$  parameter, into the Moyal one:

$$s^{(\theta)} S^{(\theta)} = 0 \quad (3.10)$$

$$(s^{(\theta)})^2 = 0 \quad (3.11)$$

It is far less obvious that the deformation of the ordinary product into the Moyal one is the only way which allows the introduction of the  $\theta$  parameter in a way which leads to an action symmetric under a nilpotent operator.

It can be verified that the action  $S^{(\theta)}$  (3.8) keeps the symmetries of its commutative counterpart  $S$ , namely it is invariant under the supersymmetry  $\delta_\mu$  (2.16)

$$\delta_\mu S^{(\theta)} = 0 , \quad (3.12)$$

and satisfies the ghost equation (2.21) as well

$$\mathcal{G}^a S^{(\theta)} = 0 . \quad (3.13)$$

It is not surprising that the above two symmetries remains unaltered although the action is non trivially modified by the introduction of  $\theta^{\mu\nu}$ . As already pointed out, the symmetry  $\delta_\mu$  is related to the topological character of the theory, which is not dismantled by  $\theta^{\mu\nu}$  which, in two dimensions, is

proportional to the Lorentz invariant Levi-Civita tensor  $\epsilon^{\mu\nu}$ . On the other hand, the identity (3.13) is well understood once we notice that in the Moyal product (3.1) the ghost field  $c^a(x)$  appears in the  $\theta$ -dependent part of the action  $S^{(\theta)}$  only differentiated, hence unaffected by the operator  $\mathcal{G}^a$ .

The superalgebra (2.17) survives in the noncommutative case

$$\begin{aligned} (s^{(\theta)})^2 &= 0 \\ \{\delta_\mu, \delta_\nu\} &= 0 \\ \{\delta_\mu, s^{(\theta)}\} &= \partial_\mu + \text{eqs of motion} \end{aligned} \quad (3.14)$$

More in detail, the last of (3.14) reads

$$\{\delta_\mu, s^{(\theta)}\}A_\nu^a = \partial_\mu A_\nu^a + \epsilon_{\mu\nu} \frac{\delta S^{(\theta)}}{\delta \phi^a} \quad (3.15)$$

$$\{\delta_\mu, s^{(\theta)}\}\phi^a = \partial_\mu \phi^a + \epsilon_{\mu\nu} \frac{\delta S^{(\theta)}}{\delta A_\nu^a} \quad (3.16)$$

$$\{\delta_\mu, s^{(\theta)}\}c^a = \partial_\mu c^a \quad (3.17)$$

$$\{\delta_\mu, s^{(\theta)}\}\bar{c}^a = \partial_\mu \bar{c}^a \quad (3.18)$$

$$\{\delta_\mu, s^{(\theta)}\}b^a = \partial_\mu b^a \quad (3.19)$$

Since the BRS operator is modified from  $s$  to  $s^{(\theta)}$ , the source term in the action is accordingly deformed, at the first order in  $\theta$ , into

$$\begin{aligned} S_{ext}^{(\theta)} &= S_{ext} + \frac{1}{2} \theta^{\alpha\beta} d^{abc} \int d^2x \left( A^{*a\mu} (\partial_\alpha A_\mu^b) (\partial_\beta c^c) \right. \\ &\quad \left. + \phi^{*a} (\partial_\alpha \phi^b) (\partial_\beta c^c) - \frac{1}{2} c^{*a} (\partial_\alpha c^b) (\partial_\beta c^c) \right) \end{aligned} \quad (3.20)$$

Summarizing, the noncommutative, two dimensional classical  $BF$  action

$$\Sigma^{(\theta)} = S_{inv}^{(\theta)} + S_{gf}^{(\theta)} + S_{ext}^{(\theta)} \quad (3.21)$$

although non trivially modified by the introduction of the noncommutative parameter  $\theta^{\mu\nu}$ , is characterized by the same set of symmetries displaying the same algebra as its commutative counterpart, namely the Slavnov-Taylor identity

$$\mathcal{S}(\Sigma^{(\theta)}) = 0, \quad (3.22)$$

the supersymmetry Ward identity

$$W_\mu \Sigma^{(\theta)} = \Delta_\mu, \quad (3.23)$$

the ghost equation

$$\mathcal{G}^a \Sigma^{(\theta)} = \Delta^a, \quad (3.24)$$

and the Landau gauge condition

$$\frac{\delta \Sigma^{(\theta)}}{\delta b^a} = \partial_\mu A^{a\mu}. \quad (3.25)$$

We stress again the non triviality of the existence of some of the above symmetries for the  $\theta$ -modified action. Particularly remarkable is the existence of a noncommutative, nilpotent, BRS symmetry  $s^{(\theta)}S^{(\theta)} = 0$  and of the supersymmetry  $\delta_\mu S^{(\theta)} = 0$ .

## 4 Stability

A necessary condition for the renormalizability of a quantum field theory is the stability of the classical action under radiative corrections. In general, stability is achieved if, after perturbing the classical action  $\Sigma$  with an infinitesimal functional with the same quantum numbers as  $\Sigma$

$$\Sigma \longrightarrow \Sigma + \epsilon \Sigma^{(c)} \quad (4.1)$$

and after imposing that the perturbed action satisfies the same set of constraints on  $\Sigma$ , the outcome is that the perturbation  $\Sigma^{(c)}$  can be reabsorbed in  $\Sigma$  through a redefinition of the fields and parameters of the theory. In particular, no new functional monomials, with respect to the classical action  $\Sigma$ , should survive this process. Otherwise, the theory would not be stable, and no renormalizations could be invoked in order to reabsorb radiative corrections. The theory would lose its predictive power and hence would not be renormalizable.

The renormalizability of the commutative, two-dimensional  $BF$  model has been proven in [14]. The supersymmetry  $\delta_\mu$  is crucial for the stability of the theory, for it prevents that the coefficients of each monomial appearing in the action depend on infinite polynomials in the scalar field  $\phi^a(x)$ :

$$\mathcal{F}(\phi) = D^{a_1 \dots a_n} \phi^{a_1} \dots \phi^{a_n} \quad (4.2)$$

where  $D^{a_1 \dots a_n}$  are completely symmetric invariant tensors. Monomials like (4.2) are gauge invariant, and do not affect power counting prescriptions, since the scalar field  $\phi^a(x)$  is dimensionless in two spacetime dimensions. Nonetheless, those infinite set of field dependent coefficients are not present in the classical action  $\Sigma$ , which would therefore be unstable and non renormalizable if the supersymmetry  $\delta_\mu$  would not occur. Indeed, the terms (4.2) are not invariant under the action of the operator  $\delta_\mu$ :

$$\delta_\mu \mathcal{F}(\phi) = n D^{a_1 \dots a_n} \varepsilon_{\mu\nu} (\partial^\nu \bar{c}^{a_1}) \phi^{a_2} \dots \phi^{a_n} \quad (4.3)$$

and hence are forbidden. On the other hand, polynomials in color singlet built with the field strength  $F_{\mu\nu}^a(x)$ , like for instance

$$\int d^2x F_{\mu\nu}^a F^{a\mu\nu} ; \int d^2x d^{abc} \tilde{F}^a \tilde{F}^b \tilde{F}^c ; \dots \quad (4.4)$$

where  $\tilde{F}^a(x) = \varepsilon^{\mu\nu} F_{\mu\nu}^a(x)$ , are invariant both under BRS and  $\delta_\mu$  symmetries, but violate power counting (notice that, in two dimensions  $\tilde{F}^a \tilde{F}^a = 2 F_{\mu\nu}^a F^{a\mu\nu}$ ).

The introduction of  $\theta^{\mu\nu}$  drastically changes this scenario. The general form of the perturbation  $\Sigma^{(c)}$  in (4.1) now takes the form

$$\Sigma^{(c)} \longrightarrow \Sigma^{(c)} + \Sigma^{(c,\theta)} \quad (4.5)$$

where

$$\Sigma^{(c,\theta)} \equiv \theta^{\mu\nu} \Sigma_{\mu\nu} + \theta^{\mu\nu} \varepsilon^{\rho\sigma} \Sigma_{\mu\nu\rho\sigma} + \theta^{\mu\nu} \varepsilon_{\mu\nu} \Sigma + \theta^{\mu\alpha} \varepsilon_{\alpha}^{\nu} \Sigma'_{\mu\nu} \quad (4.6)$$

and  $\Sigma_{\mu\nu}$ ,  $\Sigma_{\mu\nu\rho\sigma}$ ,  $\Sigma$  and  $\Sigma'_{\mu\nu}$  do not depend on the Levi-Civita tensor  $\varepsilon^{\alpha\beta}$ , and have canonical dimensions 4 and Faddeev-Popov charge 0. Now, due to the fact that, in two dimensions,  $\theta^{\mu\nu}$  is proportional to  $\varepsilon^{\mu\nu}$  (3.4), the counterterm finally reads

$$\Sigma^{(c,\theta)} = \frac{1}{\theta^2} (\varepsilon^{\mu\nu} X_{\mu\nu} + X) \quad (4.7)$$

with  $X_{\mu\nu}$  and  $X$  not depending on  $\varepsilon^{\mu\nu}$ .

The perturbed action must satisfy the same constraints as  $\Sigma^{(\theta)}$ . This implies, at the first order in the perturbation parameter  $\varepsilon$

$$\frac{\delta \Sigma^{(c,\theta)}}{\delta b^a} = 0 \quad (4.8)$$

$$\mathcal{G}^a \Sigma^{(c,\theta)} = 0 \quad (4.9)$$

$$W_{\rho} \Sigma^{(c,\theta)} = 0 \quad (4.10)$$

$$B_{\Sigma^{(\theta)}}(\Sigma^{(c,\theta)}) = 0 \quad (4.11)$$

where  $B_{\Sigma^{(\theta)}}$  is the linearized Slavnov-Taylor operator

$$B_{\Sigma^{(\theta)}} = \text{Tr} \int d^2x \left( \frac{\delta \Sigma^{(\theta)}}{\delta A^{*\mu}} \frac{\delta}{\delta A_{\mu}} + \frac{\delta \Sigma^{(\theta)}}{\delta A_{\mu}} \frac{\delta}{\delta A^{*\mu}} + \frac{\delta \Sigma^{(\theta)}}{\delta \phi^*} \frac{\delta}{\delta \phi} + \frac{\delta \Sigma^{(\theta)}}{\delta \phi} \frac{\delta}{\delta \phi^*} + \frac{\delta \Sigma^{(\theta)}}{\delta c^*} \frac{\delta}{\delta c} + \frac{\delta \Sigma^{(\theta)}}{\delta c} \frac{\delta}{\delta c^*} + b \frac{\delta}{\delta \bar{c}} \right) \quad (4.12)$$

The algebra formed by the supersymmetry Ward operators  $W_{\mu}$  (2.19) and the Slavnov-Taylor operator  $B_{\Sigma^{(\theta)}}$  (4.12) is

$$(B_{\Sigma^{(\theta)}})^2 = 0 \quad (4.13)$$

$$\{W_{\mu}, W_{\nu}\} = 0 \quad (4.14)$$

$$\{B_{\Sigma^{(\theta)}}, W_{\mu}\} = \mathcal{P}_{\mu} \equiv \sum_{\text{all fields } \Phi} \int d^2x \partial_{\mu} \Phi \frac{\delta}{\delta \Phi} \quad (4.15)$$

The first two conditions (4.8) and (4.9) are satisfied by a functional which does not depend on the Lagrange multiplier  $b^a(x)$  and depends on the undifferentiated ghost field  $c^a(x)$  at most once<sup>1</sup>. Moreover, as a consequence of the gauge condition (3.25) and of the Slavnov-taylor identity (3.22), the

<sup>1</sup>To take into account, for instance, terms like  $\int d^2x \epsilon^{\mu\nu} d^{abc} c^a \partial_{\mu} c^b \partial_{\nu} c^c$ , which satisfies, indeed, the ghost equation (4.9).

action  $\Sigma^{(\theta)}$ , like any other gauge theory, automatically satisfies an additional symmetry, called the antighost equation

$$\left( \frac{\delta}{\delta \bar{c}^a} + \partial^\mu \frac{\delta}{\delta A^{\star a\mu}} \right) \Sigma^{(\theta)} = 0 \quad (4.16)$$

which is satisfied if the fields  $A^{\star a\mu}$  and  $\bar{c}^a$  appear in the action only through the combination

$$\hat{A}^{\star a\mu} \equiv A^{\star a\mu} + \partial^\mu \bar{c}^a \quad (4.17)$$

Concerning the supersymmetry condition (4.10), its most general solution is

$$\Sigma^{(c,\theta)} = W_\mu W_\nu \Sigma^{\mu\nu}, \quad (4.18)$$

where  $\Sigma^{\mu\nu} = -\Sigma^{\nu\mu}$  is a kind of “prepotential”, in close analogy to what happens in  $N = 2$  Super Yang-Mills theory [23].

In order to show this, we develop Lorentz indices: Eq. (4.18) reads

$$\Sigma^{(c,\theta)} = 2 W_1 W_2 \Sigma^{12} \quad (4.19)$$

where

$$W_1 = \int d^2x \left( \hat{A}_2^{\star a} \frac{\delta}{\delta \phi^a} + \phi^{\star a} \frac{\delta}{\delta A_2^a} + A_1^a \frac{\delta}{\delta c^a} - c^{\star a} \frac{\delta}{\delta \hat{A}_1^{\star a}} \right) \quad (4.20)$$

$$W_2 = \int d^2x \left( -\hat{A}_1^{\star a} \frac{\delta}{\delta \phi^a} - \phi^{\star a} \frac{\delta}{\delta A_1^a} + A_2^a \frac{\delta}{\delta c^a} - c^{\star a} \frac{\delta}{\delta \hat{A}_2^{\star a}} \right) \quad (4.21)$$

Therefore, the corresponding adjoint operators read

$$W_1^\dagger = \int d^2x \left( \phi^a \frac{\delta}{\delta \hat{A}_2^{\star a}} + A_2^a \frac{\delta}{\delta \phi^{\star a}} + c^a \frac{\delta}{\delta A_1^a} - \hat{A}_1^{\star a} \frac{\delta}{\delta c^{\star a}} \right) \quad (4.22)$$

$$W_2^\dagger = \int d^2x \left( -\phi^a \frac{\delta}{\delta \hat{A}_1^{\star a}} - A_1^a \frac{\delta}{\delta \phi^{\star a}} + c^a \frac{\delta}{\delta A_2^a} - \hat{A}_2^{\star a} \frac{\delta}{\delta c^{\star a}} \right) \quad (4.23)$$

The algebra (4.14) reads

$$W_1^2 = W_2^2 = 0 ; \quad \{W_1, W_2\} = 0 \quad (4.24)$$

and

$$(W_1^\dagger)^2 = (W_2^\dagger)^2 = 0 ; \quad \{W_1, W_2^\dagger\} = \{W_2, W_1^\dagger\} = 0 \quad (4.25)$$

$$\{W_1^\dagger, W_1\} = \{W_2^\dagger, W_2\} = \sum_{\text{all fields } \Phi} \int d^2x \Phi \frac{\delta}{\delta \Phi} \equiv \mathcal{N} \quad (4.26)$$

The operator  $\mathcal{N}$  counts the number  $n_\Phi$  of fields  $\Phi$  appearing in a generic functional  $\mathcal{F}(\Phi)$

$$\mathcal{N}\mathcal{F}(\Phi) = \sum_{\text{all fields } \Phi} n_\Phi \mathcal{F}(\Phi) \equiv N \mathcal{F}(\Phi) \quad (4.27)$$

We stress that  $W_1$  and  $W_2$ , like their adjoints, are nilpotent operators with vanishing local cohomology, since all fields appear as BRS doublets [15].

Now, we are interested in the most general local integrated functional  $X$  satisfying

$$W_1 X_{(q)}^{(p)} = W_2 X_{(q)}^{(p)} = 0 \quad (4.28)$$

where  $p$  and  $q$  are mass dimension and ghost number of the functional  $X_{(q)}^{(p)}$ , respectively. Our aim is to show that any solution of Eq (4.28) can be written as follows

$$X_{(q)}^{(p)} = W_1 W_2 Y_{(q+2)}^{(p-2)} \quad (4.29)$$

Notice indeed that, from the expression (2.19), the operators  $W_\mu$  raise by one unit the mass dimension and lower by one unit the Faddeev-Popov charge.

Let us prove the result (4.29).

From (4.28) we have that

$$X_{(q)}^{(p)} = W_1 X_{(q+1)}^{(p-1)} = W_2 \widehat{X}_{(q+1)}^{(p-1)} \quad (4.30)$$

where  $X_{(q+1)}^{(p-1)}$  and  $\widehat{X}_{(q+1)}^{(p-1)}$  are generic functionals. Hence it holds

$$W_1^\dagger W_1 X_{(q+1)}^{(p-1)} = W_1^\dagger W_2 \widehat{X}_{(q+1)}^{(p-1)} \quad (4.31)$$

Therefore, using (4.26) and (4.27),

$$\mathcal{N} X_{(q+1)}^{(p-1)} - W_1 W_1^\dagger X_{(q+1)}^{(p-1)} = -W_2 W_1^\dagger \widehat{X}_{(q+1)}^{(p-1)} \quad (4.32)$$

From (4.27), we can write

$$X_{(q+1)}^{(p-1)} = \frac{1}{N} (W_1 W_1^\dagger X_{(q+1)}^{(p-1)} - W_2 W_1^\dagger \widehat{X}_{(q+1)}^{(p-1)}) \quad (4.33)$$

We then conclude that

$$X_{(q)}^{(p)} = W_1 X_{(q+1)}^{(p-1)} = W_1 W_2 Y_{(q+2)}^{(p-2)} \quad (4.34)$$

(where  $Y_{(q+2)}^{(p-2)} = -\frac{1}{N} W_2^\dagger \widehat{X}_{(q+1)}^{(p-1)}$ ), which is the desired result (4.29), or (4.19).

Let us summarize our knowledge on the prepotential  $\Sigma^{\mu\nu}$  appearing in (4.18):

- It is a local integrated functional, with canonical dimension and Faddeev-Popov charge +2;
- It may depend on the quantum fields  $A_\mu^a(x)$ ,  $\phi^a(x)$ ,  $\partial_\mu c^a(x)$ , and at most once on the undifferentiated ghost field  $c^a(x)$ ;
- It may depend on the external sources  $\widehat{A}_\mu^{\star a}(x)$ ,  $c^{\star a}(x)$ , and  $\phi^{\star a}(x)$

Making explicit the dependence on the Levi-Civita tensor  $\varepsilon_{\mu\nu}$ , as we did in (4.7), the most general candidate for  $\Sigma_{\mu\nu}$  is

$$\Sigma_{\mu\nu} = \frac{1}{\theta^2}(\varepsilon_{\mu\nu}X + X_{\mu\nu}) \quad (4.35)$$

with

$$X = \int d^2x c^a \left( \partial^2 c^b R_1^{ab}(\phi) + \partial^\mu c^b R_{2\mu}^{ab}(\phi, A) \right) \quad (4.36)$$

$$X_{\mu\nu} = \int d^2x c^a \left( \partial_\mu c^b R_\nu^{ab}(\phi, A) - \partial_\nu c^b R_\mu^{ab}(\phi, A) \right) \quad (4.37)$$

where  $R_1^{ab}(\phi)$ ,  $R_{2\mu}^{ab}(\phi, A)$  and  $R_\mu^{ab}(\phi, A)$  are generic functions.

Since, due to (4.8),  $X_{\mu\nu}$  and  $X$  are independent from the Lagrange multiplier  $b^a(x)$ , recalling (2.21), the ghost equation condition (4.9) reads

$$\mathcal{G}^a X = \int d^2x \frac{\delta}{\delta c^a} X = 0 \quad (4.38)$$

$$\mathcal{G}^a X_{\mu\nu} = \int d^2x \frac{\delta}{\delta c^a} X_{\mu\nu} = 0 \quad (4.39)$$

which give, respectively

$$\int d^2x \partial^\mu c^b (-\partial_\mu R_1^{ab}(\phi) + R_{2\mu}^{ab}(\phi, A)) = 0 \quad (4.40)$$

$$\int d^2x c^b (-\partial_\mu R_\nu^{ab}(\phi, A) + \partial_\nu R_\mu^{ab}(\phi, A)) = 0 \quad (4.41)$$

which are satisfied if

$$R_{2\mu}^{ab}(\phi, A) = \partial_\mu R_1^{ab}(\phi) \quad (4.42)$$

$$R_\mu^{ab}(\phi, A) = \partial_\mu R^{ab}(\phi) \quad (4.43)$$

where again  $R^{ab}(\phi)$  is a generic function. Notice that no dependence on the gauge field  $A_\mu^a(x)$  is allowed in the prepotential  $\Sigma_{\mu\nu}$ , reminding again  $N = 2$  Super Yang-Mills theory [23]. Hence, we have

$$X = \int d^2x \partial^\mu c^a \partial_\mu c^b R_1^{ab}(\phi) \quad (4.44)$$

$$X_{\mu\nu} = \int d^2x \partial_\mu c^a \partial_\nu c^b R^{ab}(\phi) \quad (4.45)$$

where  $R_1^{ab}(\phi)$  and  $R^{ab}(\phi)$  (which we rename  $R_2^{ab}(\phi)$ ) are generic functions of the scalar field  $\phi^a(x)$ , and  $R_1^{ab}(\phi) = -R_1^{ba}(\phi)$ ,  $R_2^{ab}(\phi) = R_2^{ba}(\phi)$ .

Therefore, our candidate for the  $O(\theta)$  counterterm is

$$\Sigma^{(c,\theta)} = W_\mu W_\nu \int d^2x \frac{1}{\theta^2} \left( \varepsilon^{\mu\nu} \partial^\rho c^a \partial_\rho c^b R_1^{ab}(\phi) + \partial^\mu c^a \partial^\nu c^b R_2^{ab}(\phi) \right) \quad (4.46)$$

On integrated local functionals  $\int d^2x F(\Phi)$ , the algebraic relation (4.15) reads

$$\{B_{\Sigma^{(\theta)}}, W_\mu\} \int d^2x F(\Phi) = \mathcal{P}_\mu \int d^2x F(\Phi) = \int d^2x \partial_\mu F(\Phi) = 0 \quad (4.47)$$

and the Slavnov-Taylor constraint (4.11) becomes <sup>2</sup>

$$B_{\Sigma^{(\theta)}} \Sigma_{\mu\nu} = s \Sigma_{\mu\nu} = 0 \quad (4.48)$$

Therefore, it must be

$$s \int d^2x \left( \varepsilon^{\mu\nu} \partial^\rho c^a \partial_\rho c^b R_1^{ab}(\phi) + \partial^\mu c^a \partial^\nu c^b R_2^{ab}(\phi) \right) = 0 \quad (4.49)$$

and it is easily seen that this constraint is satisfied if

$$s R_1^{ab}(\phi) = f^{amn} c^m R_1^{bn}(\phi) - f^{bm n} c^m R_1^{an}(\phi) \quad (4.50)$$

$$s R_2^{ab}(\phi) = -f^{amn} c^m R_2^{bn}(\phi) - f^{bm n} c^m R_2^{an}(\phi) \quad (4.51)$$

Locality imposes that  $R_1^{ab}(\phi)$  and  $R_2^{ab}(\phi)$  are not really generic functions, but polynomials in  $\phi^a(x)$ . Taking into account the symmetries in the color indices  $a$  and  $b$ , we can write

$$\begin{aligned} R_1^{ab}(\phi) &= \alpha f^{abc} \phi^c + O(\phi^2) = \sum_{n=1}^{\infty} T_1^{[ab]a_1..a_n} \phi^{a_1}..\phi^{a_n} \\ &= \sum_{n=1}^{\infty} T_1^{[ab]P_n} \Phi^{P_n} \end{aligned} \quad (4.52)$$

$$\begin{aligned} R_2^{ab}(\phi) &= \beta \delta^{ab} + \gamma d^{abc} \phi^c + O(\phi^2) = \sum_{n=0}^{\infty} T_2^{(ab)a_1..a_n} \phi^{a_1}..\phi^{a_n} \\ &= \sum_{n=0}^{\infty} T_2^{(ab)P_n} \Phi^{P_n} \end{aligned} \quad (4.53)$$

where  $\alpha, \beta$  and  $\gamma$  are constants, and we adopted the short-hand notation  $P_n \equiv (p_1 \dots p_n)$ . More in general, rigid gauge invariance requires that  $T_{1,2}^{abP_n}$  are constant invariant tensors [15], built from the Kronecker  $\delta^{ab}$ , the gauge group structure constants  $f^{abc}$  and the completely symmetric rank three tensor  $d^{abc}$ . Therefore, conditions (4.50) and (4.51) are automatically satisfied, and the most general counterterm, satisfying all the constraints (4.8) - (4.11), is

$$\begin{aligned} \Sigma^{(c,\theta)} &= W_\mu W_\nu \int d^2x \frac{1}{\theta^2} \left( \sum_{n=1}^{\infty} T_1^{[ab]P_n} \varepsilon^{\mu\nu} c^{a\rho} c_\rho^b \Phi^{P_n} + \sum_{n=0}^{\infty} T_2^{(ab)P_n} c^{a\mu} c^{b\nu} \Phi^{P_n} \right) \\ &\equiv \Sigma_1^{(c,\theta)} + \Sigma_2^{(c,\theta)} \end{aligned} \quad (4.54)$$

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<sup>2</sup>The prepotential  $\Sigma_{\mu\nu}$  does not depend on the external sources, and the action of the linearized Slavnov-Taylor operator (4.12) on the quantum fields coincides with that of the BRS operator  $s$ , since  $B_{\Sigma^{(\theta)}} \Phi = \frac{\delta \Sigma^{(\theta)}}{\delta \Phi^*} = s\Phi$ , where  $\Phi$  is a generic quantum field  $\Phi = (A, c, \phi)$ .

with

$$\begin{aligned} \Sigma_1^{(c,\theta)} &= \sum_{n=1}^{\infty} \int d^2x T_1^{[ab](a_1..a_n)} \left[ 4\partial^\mu \phi^{\star a} c_\mu^b \phi^{a_1} \phi^{a_2} + 2\varepsilon^{\mu\nu} \partial^\rho A_\nu^a \partial_\rho A_\mu^b \phi^{a_1} \phi^{a_2} \right. \\ &\quad \left. - 4n\partial^\mu A^{a\nu} c_\mu^b \hat{A}_\nu^{\star a_1} \phi^{a_2} + 4nc^{\alpha\mu} c_\mu^b c^{\star a_1} \phi^{a_2} \right. \\ &\quad \left. + \varepsilon^{\mu\nu} n(n-1) c^{a\rho} c_\rho^b \hat{A}_\mu^{\star a_1} \hat{A}_\nu^{\star a_2} \right] \phi^{a_3}.. \phi^{a_n} \end{aligned} \quad (4.55)$$

$$\begin{aligned} \Sigma_2^{(c,\theta)} &= \sum_{n=0}^{\infty} \int d^2x T_2^{(ab)(a_1..a_n)} \left[ 2\varepsilon^{\mu\nu} \partial_\mu \phi^{\star a} c_\nu^b \phi^{a_1} \phi^{a_2} + \partial^\mu A^{a\nu} \partial_\nu A_\mu^b \phi^{a_1} \phi^{a_2} \right. \\ &\quad \left. - \partial A^a \partial A^b \phi^{a_1} \phi^{a_2} - 2n\varepsilon^{\mu\alpha} \partial_\mu A_\nu^a c^{\nu b} \hat{A}_\alpha^{\star a_1} \phi^{a_2} + 2n\varepsilon^{\mu\alpha} c_\mu^a \partial A^b \hat{A}_\alpha^{\star a_1} \phi^{a_2} \right. \\ &\quad \left. + n\varepsilon^{\mu\nu} c_\mu^a c_\nu^b c^{\star a_1} \phi^{a_2} + n(n-1) c^{a\mu} c^{\nu b} \hat{A}_\mu^{\star a_1} \hat{A}_\nu^{\star a_2} \right] \phi^{a_3}.. \phi^{a_n} \end{aligned} \quad (4.56)$$

where  $c_\mu^a \equiv \partial_\mu c^a$ . We stress that the counterterm (4.54) does not belong to the integrated cohomology of the Slavnov-Taylor operator  $B_{\Sigma^{(\theta)}}$ , since it can be written as an exact BRS cocycle:

$$\Sigma^{(c,\theta)} = B_{\Sigma^{(\theta)}} Z \quad (4.57)$$

with

$$Z = W_\mu W_\nu \int d^2x \frac{1}{\theta^2} \left( \sum_{n=1}^{\infty} \varepsilon^{\mu\nu} T_1^{[ab]P_n} A^{a\rho} c_\rho^b + \sum_{n=0}^{\infty} T_2^{(ab)P_n} A^{a\mu} c^{b\nu} \right) \Phi^{P_n} \quad (4.58)$$

The counterterm  $\Sigma^{(c,\theta)}$  represents the quantum corrections of the classical action  $\Sigma^{(\theta)}$  (3.21), whose  $\theta$ -dependent part is

$$\begin{aligned} \Sigma^{(\theta)} \Big|_{O(\theta)} &= \frac{1}{2} d^{abc} \theta^{\alpha\beta} \int d^2x \left( -\frac{1}{2} \varepsilon^{\mu\nu} \phi^a (\partial_\alpha A_\mu^b) (\partial_\beta A_\nu^c) + (\partial_\mu \bar{c}^a) (\partial_\alpha A_\mu^b) (\partial_\beta c^c) + \right. \\ &\quad \left. A^{*\mu a} (\partial_\alpha A_\mu^b) (\partial_\beta c^c) + \phi^{*a} (\partial_\alpha \phi^b) (\partial_\beta c^c) - \frac{1}{2} c^{*a} (\partial_\alpha c^b) (\partial_\beta c^c) \right) \end{aligned} \quad (4.59)$$

The theory is stable under radiative corrections if the most general counterterm can be reabsorbed by a redefinition of fields and parameters of the classical theory. In the commutative case, the two dimensional BF model is finite, namely no local counterterm is compatible with the constraints (4.8) - (4.11) and there is nothing to be reabsorbed, or, in other words, no renormalizations of fields or parameters are present. On the contrary, it is apparent from the direct comparison of the two terms (4.55) and (4.56) forming the counterterm and the  $\theta$ -dependent part of the classical action (4.59), that such reabsorption is, in the noncommutative case, impossible. The theory is highly unstable, and radiative corrections are out of control.

In order that the theory is renormalizable, it is not difficult to see that the counterterm, instead of consisting in a double infinity of terms, should collapse into one term only

$$T_1^{[ab]a_1..a_n} = 0 \quad \forall n \quad (4.60)$$

and

$$T_2^{(ab)a_1..a_n} = \begin{cases} 0 & \forall n \neq 1 \\ \frac{1}{4}d^{aba_1} & \text{if } n = 1 \end{cases} \quad (4.61)$$

This counterterm might be reabsorbed by a renormalization of the  $\theta$  parameter. But this is not the case, and the theory, due to the infinite terms present in the radiative corrections, has no predictive power. Even before asking the question of the presence of noncommutative anomalies (the commutative case is of course anomaly-free [14]), we must conclude that the noncommutative theory, taken just as a quantum field theory, is highly non renormalizable. We believe that this illness is not peculiar of the two dimensional  $BF$  model, but it is due to the simple recipe of constructing noncommutative theories from their commutative counterpart just substituting the ordinary product between fields with the groenewold-Moyal one.

## 5 Anomaly

As it is well known, anomalies are quantum breakings of symmetries. Let  $\Gamma$  be the functional generator of 1PI Green functions, or, equivalently, quantum vertex, quantum or effective action. It holds

$$\Gamma = \Sigma^{(\theta)} + O(\hbar) \quad (5.1)$$

where  $\Sigma^{(\theta)}$  is the classical action (3.21). The constraints (3.23), (3.24), (3.25) and (4.16), being linear, are easily shown to hold also at the quantum level:

1. gauge condition

$$\frac{\delta \Gamma}{\delta b^a(x)} = \partial A^a \quad (5.2)$$

2. antighost equation

$$\left( \frac{\delta}{\delta A^{\star a}(x)} + \frac{\delta}{\delta \partial_\mu \bar{c}^a(x)} \right) \Gamma = 0 \quad (5.3)$$

3. ghost equation

$$\mathcal{G}^a \Gamma = \Delta^a \quad (5.4)$$

4. supersymmetry

$$W_\mu \Gamma = \Delta_\mu \quad (5.5)$$

The only symmetry whose quantum extension should be handled with care is the (nonlinear) Slavnov-Taylor identity (3.22), which *a priori* is broken

$$\mathcal{S}(\Gamma) = \mathcal{A} \quad (5.6)$$

by a quantum breaking  $\mathcal{A}$  which, according to the Quantum Action Principle [7, 8, 9, 10, 11], at the lowest nonvanishing order in  $\hbar$ , is a local integrated functional with canonical dimensions 2 and Faddeev-Popov charge +1

$$\mathcal{A} = \Delta_{(1)}^{(2)} + O(\hbar \Delta_{(1)}^{(2)}) = \int d^2x \Delta_{(1)}^{(2)}(x) + O(\hbar \Delta_{(1)}^{(2)}) \quad (5.7)$$

The web of algebraic relations [14] results in the following consistency conditions on  $\Delta_{(1)}^{(2)}$ :

$$\frac{\delta \Delta_{(1)}^{(2)}}{\delta b^a(x)} = 0 ; \quad (5.8)$$

$$\left( \frac{\delta}{\delta A_\mu^{\star a}(x)} + \frac{\delta}{\delta \partial_\mu \bar{c}^a(x)} \right) \Delta_{(1)}^{(2)} = 0 ; \quad (5.9)$$

$$\mathcal{G}^a \Delta_{(1)}^{(2)} = 0 ; \quad (5.10)$$

$$W_\mu \Delta_{(1)}^{(2)} = 0 ; \quad (5.11)$$

$$B_{\Sigma^{(\theta)}} \Delta_{(1)}^{(2)} = 0 . \quad (5.12)$$

The last constraint (5.12) is the cohomology problem commonly known as Wess-Zumino consistency condition [24]. The constraints (5.8)-(5.12) are basically the same that we have already imposed on the counterterm  $\Sigma^{(c,\theta)}$ , with the difference that the functional  $\Delta_{(1)}^{(2)}$  belongs to the Faddeev-Popov sector with charge +1 instead of zero.

We already know from the previous section that the solution of the first four constraints is a functional  $\Delta_{(1)}^{(2)}$  which does not depend on the Lagrange multiplier  $b^a(x)$ , depends on the antighost field  $\bar{c}^a(x)$  and on the external source  $A_\mu^{\star a}(x)$  only through the combination  $\hat{A}_\mu^{\star a}(x)$  (4.17), it may depend on the undifferentiated ghost field  $c^a(x)$  at most once and it can be written as

$$\Delta_{(1)}^{(2)} = W_\mu W_\nu \Delta_{(3)}^{(0)\mu\nu} , \quad (5.13)$$

where  $\Delta_{(3)}^{(0)\mu\nu}$  is a local integrated functional with dimensions zero and Faddeev-Popov charge +3. The only possible  $\theta$ -independent term is

$$\Delta_{(3)}^{(0)\mu\nu} = \int d^2x \varepsilon^{\mu\nu} c^a c^b c^c R^{[abc]}(\phi) \quad (5.14)$$

with  $R^{[abc]}(\phi)$  polynomial in  $\phi^a(x)$  and antisymmetric in the color indices  $abc$ . But this only commutative candidate for the anomaly is ruled out by the ghost equation (5.10), and, as we already knew [14], the commutative theory is anomaly free.

Let us see if the situation is different at the order  $\theta$ . Recalling that  $\theta^{\mu\nu} \propto \varepsilon^{\mu\nu}$ , we must look for anomalies whose structure is

$$\Delta_{(1)}^{(2)} = W_\mu W_\nu \frac{1}{\theta^2} \left( \varepsilon^{\mu\nu} \Delta_{(3)}^{(2)} + \Delta_{(1)}^{(2)\mu\nu} \right) . \quad (5.15)$$

The only possibilities for functionals depending on the undifferentiated ghost field  $c^a(x)$  at most once, are

$$\Delta_{(3)}^{(2)} = \int d^2x \, c^a \partial^\rho c^b \partial_\rho c^c \, R_1^{a[bc]}(\phi) \quad (5.16)$$

$$\Delta_{(3)}^{(2)\mu\nu} = \int d^2x \, c^a \partial^\mu c^b \partial^\nu c^c \, R_2^{a(bc)}(\phi) , \quad (5.17)$$

with  $R_1^{a[bc]}(\phi)$  and  $R_2^{a(bc)}(\phi)$  polynomials in  $\phi^a(x)$ .

The ghost equation (5.10) is satisfied if

$$R_1^{a[bc]}(\phi) = 0 \quad (5.18)$$

$$R_2^{a(bc)}(\phi) = \alpha d^{abc} , \quad (5.19)$$

where  $\alpha$  is a constant. Hence the only candidate for the  $O(\theta)$  anomaly is

$$\Delta_{(1)}^{(2)} = W_\mu W_\nu \int d^2x \, \frac{1}{\theta^2} \alpha d^{abc} c^a \partial^\mu c^b \partial^\nu c^c , \quad (5.20)$$

but  $\Delta_{(1)}^{(2)}$  is not BRS invariant, hence does not satisfy the Wess-Zumino consistency condition (5.12). We conclude that, at least at the first order in  $\theta$ , non commutativity does not introduce anomalies, but nevertheless the noncommutative theory is still not renormalizable.

## 6 Conclusions

In this paper we considered the noncommutative deformation of the two dimensional BF model, which, taken as an ordinary commutative theory, is topological, and it shares with all other commutative, topological field theories the property of finiteness. This means that the topological, commutative BF model has a particularly simple quantum extension, since, as it has been shown in [14], no local counterterm turns out to be compatible with the symmetries characterizing the classical theory. In the noncommutative extension of quantum field theories, a new parameter comes into play,  $\theta_{\mu\nu}$ , which has mass dimension minus two. The easy recipe to treat a field theory living in the noncommutative plane is that of describing it through an ordinary, commutative field theory, where the product between quantum fields is substituted by the groenewold-Moyal star product (3.2). This is the so called “Moyal prescription” [17, 16]. Now, as it is widely known, the product between quantum fields is one of the delicacies of quantum field theory, since distributions at coinciding points are not well defined. The Moyal prescription just modifies the product between quantum fields, and we believe that serious attention should be paid to such an operation. We showed, in the simple example of this finite two dimensional theory, that the Moyal prescription leads to a meaningless quantum field theory, and we believe that the situation could be even worse in more complicated quantum field theories, living

in higher dimensional spacetime. We do not conclude that in general the noncommutative theories do not make sense. Our milder claim is that one should be careful in applying to them the usual rules of *quantum field theories*, in the sense that we are going to explain. The main feature of a quantum field theory is *locality*. Locality requires that the action of a quantum field theory is an integrated local functional, moreover, such quantum functional (namely the classical action and its quantum radiative corrections) must be *analytical* in all parameters (masses, coupling constants, ...) appearing in the theory. We limited ourselves just to the analytical sector in  $\theta$ , aware of the fact that the analytical sector does not include the whole noncommutative theory. Non commutativity indeed may open a non analytical sector in which the ordinary rules of quantum fields theory do not apply. Nevertheless, the analytical sector, in which a  $\theta$ -expansion is allowed, does exist, and it is precisely this which we considered, limiting even more ourselves to the first order in  $\theta_{\mu\nu}$ . On non-analiticity introduced by the noncommutative deformation of quantum field theories, see for instance [16, 5].

Summarizing, our playground is quite restricted: it is the noncommutative theory taken in its quantum field theoretical sector, which must be local and analytical in all its parameters, including  $\theta_{\mu\nu}$ , and we looked to the first order in its  $\theta$ -expansion. But, even within this fence, we found that the quantum corrections are represented by a double infinity of terms, (4.54), which cannot be reabsorbed, as they should for a renormalizable quantum field theory, by a redefinition of fields and parameters of the theory. Therefore, the model is not renormalizable in the sense that it loses any predictive power. This occurrence should be interpreted, in our opinion, as a signal that there must be some novel feature in the noncommutative model in order to make it sensible. For instance, the two families of couplings could be a Taylor expansion of some functions; but this needs a precise relation between the coupling constants, relation which is not enforced by the known symmetries. Is there some new symmetry brought about by noncommutativity?

At this point, it is useless to go to higher orders in the  $\theta$ -expansion, since we can already conclude that the noncommutative theory does not exist as a quantum field theory. It is just something else. We believe that the situation might be more dramatic in higher dimensions, where the Landau gauge is not a compulsory choice, as it is in two and three dimensions, where the gauge parameter is massive, and gauge choices else than the Landau one lead to severe infrared problems. The Landau gauge theory is characterized by the ghost equation condition (5.4), which represents a very strong constraint on radiative corrections [22]. The double infinities of instabilities (4.54) would be much more without the protection of this constraint. On the other hand, at first order in  $\theta_{\mu\nu}$ , we did not find any noncommutative anomaly. This reinforces the impression that the instability problem could be overcome by adding some constraints which are peculiar of the noncommutative model.

We are very curious to investigate whether a  $\theta$ -dependent anomaly might

occur at higher orders and/or in different theories, like for instance four dimensional Yang-Mills theory, where we expect that, besides the obvious non-commutative extension of the Adler-Bardeen-Jackiw anomaly, other, purely noncommutative, anomalies exist.

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